6.1 Covariance Matrix

The log likelihood is

$$\mathcal{L} = N \log \sigma - \frac{1}{2\sigma^2} \sum_i (X_i - \mu)^2$$

and so the elements of the Hessian matrix are

$$\frac{\partial^2 \mu}{\partial \mu^2} = \frac{N}{\sigma^2}$$

$$\frac{\partial^2 \sigma}{\partial \sigma^2} = \frac{3}{\sigma^4} \sum (X_i - \mu)^2 + \frac{N}{\sigma^2}$$

and

$$\frac{\partial^2 \mu}{\partial \mu \partial \sigma} = \frac{2}{\sigma^2} \sum (X_i - \mu).$$

Taking expectations over many realizations of the data X_i we get

$$\frac{\partial^2 \sigma}{\partial \sigma^2} = \frac{4N}{\sigma^2}$$

and

$$\frac{\partial^2 \mu}{\partial \mu \partial \sigma} = 0.$$

The covariance matrix is thus

$$C = \begin{bmatrix} \frac{\sigma^2}{N} & 0 \\ 0 & \frac{\sigma^2}{4N} \end{bmatrix}.$$

The estimated (population) mean and variance are uncorrelated.